Fields in AdS / Operators in CFT

jueves, 24 de noviembre de 2022 12:57

$$ds^2 = \frac{dz^2 - dt^2 + d\bar{x}_{d-1}^2}{z^2} \qquad \text{units } L = 1$$

Massive scalar field in AdS_{d+1}

$$(\Box - m^2)\,\phi(t, z, \bar{x}) = 0$$

$$\phi(t, z, \bar{x}) = e^{-i\omega t + i\bar{k}\cdot\bar{x}}\,\phi_{\omega \bar{k}}(z)$$

$$\Box = \frac{1}{\sqrt{-g}}\partial_a \sqrt{-g}\, g^{ab}\partial_b \qquad \sqrt{-g} = z^{-d-1}$$

$$= z^2(\omega^2 - \bar{k}^2) + z^{d+1}\partial_z z^{-d+1}\partial_z$$

$$(\Box - m^2)\phi = 0 \Rightarrow \left[z^2(\omega^2 - \bar{k}^2) - m^2 + z^2\partial_z^2 - (d-1)\,z\partial_z\right]\phi_{\omega\bar{k}}(z) = 0$$

Bessel equation, can be solved explicitly.
But we will also be interested in geometries which are only asymptotic to AdS.

Study asymptotics of field near boundary $z \simeq 0$

$$(z^2\partial_z^2 - (d-1)\,z\partial_z - m^2)\,\phi_{\omega\bar{k}}(z) \simeq 0 \quad \Rightarrow \phi \sim z^{\Delta}$$

$$\Delta(\Delta - 1) - (d-1)\Delta - m^2 = 0$$

Two roots: $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$

and $d - \Delta = \frac{d}{2} - \sqrt{\frac{d^2}{4} + m^2}$

$$\phi_{\omega\bar{k}} = \underbrace{\phi^{(0)}_{\omega\bar{k}}\, z^{d-\Delta}}_{\text{growing as } z\to 0} + \underbrace{\phi^{(d)}_{\omega\bar{k}}\, z^{\Delta}}_{\text{decaying as } z\to 0}$$

For $m^2 > 0$

$\Delta > 0$

$d - \Delta < 0$

$d - \Delta$: non-normalizable mode (with KG product)

Δ : normalizable mode

Non-normalizable modes are not dynamical: They require ∞ energy To get excited. Instead, we consider Them as <u>fixed</u> by specifying Dirichlet bc's, ie fixing Their amplitude $\phi^{(0)}$ To a specified value.

Then we impose another bc in The bulk, Typically regularity at $z = \infty$ or at a horizon in The bulk, if iT's an AdS black hole. This Then fixes The value of $\phi^{(d)}(x)$ for a given boundary value $\phi^{(0)}(x)$

We will Think of <u>$\phi^{(0)}(x)$ as a source</u> (non-dynamical) at large distance, and <u>$\phi^{(d)}(x)$ as The</u> (dynamical) <u>field response</u> To The source.

Think, eg. of Tidal deformations in gravity.

We can relate This To conformal field operators:

Recall That $x^\mu \to \lambda x^\mu$, $z \to \lambda z$ is an AdS isometry which acts as dilatation at The bdry

For a mode w/ exponent Δ

$$\phi(z, x^\mu) \longrightarrow \phi(\lambda z, \lambda x^\mu) \sim \lambda^\Delta \phi(z, x^\mu)$$

Transforms as operator $\mathcal{O}_\Delta(x^\mu)$ of conformal dimension Δ

(eg for Maxwell's Theory, we have a scalar operator given by $F_{\mu\nu}(x)F^{\mu\nu}(x)$, with dimension $\Delta = d$)

In a CFT, we can deform the theory by a source $J(x)$

$$I \to I_{CFT} - \int d^d x \, J(x)\, \mathcal{O}(x) \quad : \; J \sim \text{space-dependent coupling}$$

In the path integral, this gives the generator of correlation fns

$$Z[J] = \int \mathcal{D}\Phi \; e^{-I_{CFT} + \int J \mathcal{O}}$$

Then

$$\left.\frac{\delta Z}{\delta J(x)}\right|_{J=0} = \int \mathcal{D}\Phi \, \mathcal{O}(x)\, e^{-I_{CFT}} = \langle \mathcal{O}(x) \rangle$$

and $$\left.\frac{\delta^2 Z}{\delta J(x)\,\delta J(y)}\right|_{J=0} = \langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle$$

so we can compute the propagator (after Wick rotn)

J are fixed sources, and in $\int d^d x \, J(x)\, \mathcal{O}_\Delta(x)$ the (mass) dimension of J is $d-\Delta$ so $J\mathcal{O}_\Delta$ has dim $= d$

This is like we had with $A \to d-\Delta$
$B \to \Delta$

So we think of non-normalizable mode as the source $\phi^{(0)}(x) = J(x)$

and of normalizable mode as the field's vev response to it $\quad \phi^{(d)}(x) = \langle \mathcal{O}_\Delta(x) \rangle$

So we have a correspondence

$$\phi(x^\mu, z) \longleftrightarrow \text{scalar operator } \mathcal{O}_\Delta(x^\mu)$$

$$\phi^{(0)}(x^\mu) \longleftrightarrow \text{source } J(x)$$

$\phi^{(d)}(x^\mu) \quad \longleftrightarrow \quad$ vev $\langle O_\Delta(x) \rangle$

$m^2 \quad \longleftrightarrow \quad$ dimension $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$

Note:

- if $\Delta \in \mathbb{R} \Rightarrow m^2 \geq -\frac{d^2}{4} \equiv m^2_{BF}$ Breitenlohner-Freedman bound

  $\left(-\frac{d^2}{4} \leq m^2 < 0 \text{ is allowed} \right)$

  In gravity this means that solutions decay

  It implies the CFT unitarity bound $\Delta \geq \frac{d}{2} - 1$ for a scalar operator, for both Δ and $d - \Delta$

- for $m^2_{BF} < m^2 < m^2_{BF} + 1$ both modes are normalizable

  Can choose which one to fix (alternate quantization)

- $\Delta = d$: marginal operator

  $m^2 = 0$ eg F^2 in Maxwell; adding it corresponds to $e \to e + \delta e$

  We have $\phi(x,z) = \phi^{(0)}(x) + \dots + z^d \phi^{(d)}(x)$

  The field approaches a const at $z=0$

  $\Delta < d$: relevant operator

  $m^2 < 0$ $\quad \phi(x,z) = z^{d-\Delta} \phi^{(0)}(x) + \dots$

  vanishes at UV $z \to 0$

  $\Delta > d$: irrelevant operator

  $m^2 > 0$ blows up at UV $z \to 0$