Divergences and holographic renormalization

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We have seen that

- short distance in QFT $\leftrightarrow$ large distance in AdS
- energy scale of QFT $\leftrightarrow$ radial coordinate in AdS

We have a local QFT definition at short distances:

"CFT lives at the boundary of AdS"

but it is everywhere in the bulk, at different energy scales

Short-distance structure (UV) of CFT corresponds to asymptotic behavior of AdS gravity

<u>QFTs have divergences at short distances</u>

Renormalization: regularization + subtraction of UV divergences

Regularization: short-distance cutoff

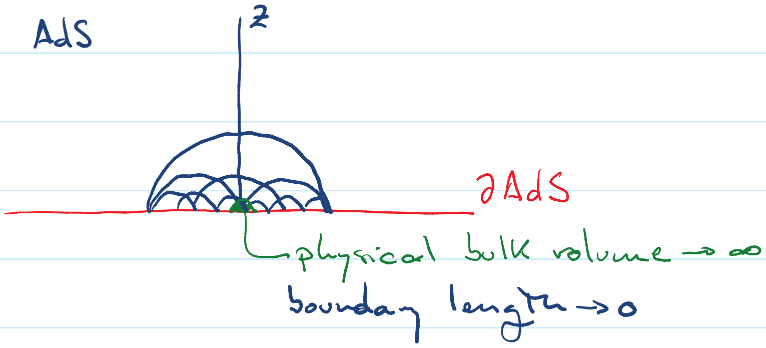
Subtraction: "local counterterms"

<u>Gravity has divergences at large distances</u>, from infinite volume

Regularization: long-distance radial cutoff

Subtraction: we have performed "background subtraction"

but AdS/CFT says that one can also eliminate divergences by introducing "local boundary counterterms" "local boundary counterterms"



We can see this in the Euclidean action

$$I = -\frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{g}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{8\pi G}\int_{\partial M} d^d x\sqrt{h}\, K$$

If we evaluate this in empty AdS_{d+1}

$$ds^2 = \left(\frac{r^2}{L^2}+1\right)d\tau^2 + \frac{dr^2}{\frac{r^2}{L^2}+1} + r^2 d\Omega_{d-1}$$

we find it is divergent.

with a cutoff at $r = R$

For the EH term use that for an AdS solution

$$R_{\mu\nu} = -\frac{d}{L^2}g_{\mu\nu} \qquad R = -\frac{d(d+1)}{L^2}$$

so $$I_{EH} = -\frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{g}\,\frac{-2d}{L^2} = \frac{d}{8\pi G L^2}\int d\tau\, d\Omega_{d-1} r^{d-1}$$

$$= \frac{\beta\Omega_{d-1}}{8\pi G L^2}R^d$$

For GHY use that $\sqrt{h}K = n^a\partial_a\sqrt{h}$

with $n^r = \sqrt{\frac{r^2}{L^2}+1}$ and $\sqrt{h} = \sqrt{\frac{r^2}{L^2}+1}\, r^{d-1}\,\Omega_{d-1}$

so $$\sqrt{h}K\Big|_{r=R} = \sqrt{\frac{r^2}{L^2}+1}\,\partial_r\left(r^{d-1}\sqrt{\frac{r^2}{L^2}+1}\right)\Bigg|_{r=R}\Omega_{d-1}$$

$$\left[\text{so} \quad \sqrt{h}\,K\Big|_{r=R} = \sqrt{\frac{r^2}{L^2}+1}\;\partial_r\left(r^{d-1}\sqrt{\frac{r^2}{L^2}+1}\right)\Big|_{r=R}\,\Omega_{d-1} \right]$$

Then

$$I_{EH} = \frac{\beta\,\Omega_{d-1}}{8\pi G L^2} R^d$$

$$I_{GHY} = \frac{\beta\,\Omega_{d-1}}{8\pi G L^2} R^d \left(-d-(d-1)\frac{L^2}{R^2}\right)$$

We want to subtract the divergences by adding counterterms to the action:

$$I_{CT} = \frac{1}{8\pi G}\int_{\partial M} d^dx \sqrt{h}\; F(L, \mathcal{R}, \nabla\mathcal{R})$$

boundary terms which are local, constructed out of the boundary metric and its intrinsic curvature tensor $\mathcal{R}_{abcd}$, ie $\mathcal{R}$, $\mathcal{R}_{ab}$, etc

From the structure of the quantum effective action of a QFT, we expect that

$$I_{ct} = \frac{1}{8\pi G L}\int_{\partial M} d^dx\sqrt{h}\left(c_0 + c_1 L^2\mathcal{R} + c_2 L^4\,(\text{curvature}^2) + \dots\right)$$

where $c_0, c_1, \dots$ are d-dependent pure numbers.

The counterterms are divergent, since

$$ds^2 = \frac{dz^2 + \gamma_{\mu\nu}dx^\mu dx^\nu}{z^2} + \dots \qquad z = 1/r$$

$$= \frac{dr^2}{r^2} + r^2\gamma_{\mu\nu}dx^\mu dx^\nu + \dots$$

at $r = R$ $\quad h_{\mu\nu} = R^2\gamma_{\mu\nu}$ $\qquad$ $\gamma_{\mu\nu}$ is the finite CFT metric

$$\sqrt{h} = R^d\sqrt{\gamma}$$

More frequently one takes the cutoff as $z = \varepsilon \ll 1$
and $R = 1/\varepsilon$

So $\sqrt{h} \sim \varepsilon^{-d}$

and curvature terms are $\sqrt{h}\,\mathcal{R} \sim \varepsilon^{-d+2}$
(on dimensional grounds) $\sqrt{h}\,(\text{curv})^2 \sim \varepsilon^{-d+4}$

so for $d=2$ we only need $\frac{1}{8\pi G L}\int_{\partial M} c_0 \sqrt{h}$

and for $d \leq 4$ we do with $\frac{1}{8\pi G L}\int_{\partial M} \sqrt{h}\,(c_0 + c_1 L^2 \mathcal{R})$

We can easily fix $c_0 = d-1$
$c_1 = \frac{1}{2(d-2)}$

so as to cancel the divergences of empty AdS_5.

(There's a more systematic, recursive way of deriving the counterterms, based on the Fefferman-Graham expansion)

So then

not a counterterm

$$I = -\frac{1}{16\pi G}\int_M d^{d+1}x\,\sqrt{g}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{8\pi G}\int_{\partial M} d^d x\,\sqrt{h}\,K$$

$$+ \frac{1}{16\pi G L}\int_{\partial M} d^d x\,\sqrt{h}\left(d-1 + \frac{L^2}{2(d-2)}\mathcal{R}\right)$$

and for empty AdS_{d+1}

AdS_3 $\quad I = -\frac{\beta}{8G}$

AdS_4 $\quad I = 0$

AdS_5 $\quad I = \beta \frac{3\pi}{32G} L^2$

If $I = \beta F = \beta E - S$ we see that in AdS_3, AdS_5 there's a β-independent, vacuum energy E_0.
These are the Casimir energies of the CFT_2, CFT_4 in $\mathbb{R}_t \times S^1$ and $\mathbb{R}_t \times S^3$
These terms couldn't have been obtained with background subtraction.

Notice that it is essential to have the AdS length scale in order to write these terms as local invariants
In AF gravity, a counterterm action can't be local.
It involves, eg, $\sqrt{R}$.
This is one of the suggestions that a holographic quantum theory of AF gravity must be a non-local theory
(another piece of evidence comes from the RT formula)